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Approximation Properties of Recursively Defined Bernstein-Type Operators*

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We consider a generalization of classical Bernstein operators obtained by replacing the binomial coefficients with general ones satisfying a suitable recursive relation. We study the uniform convergence of these operators together with some quantitative estimates and regularity properties. Finally, in some particular cases, we investigate the behavior of the iterates. © 1996 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARY RESULTS

This paper takes its motivation from the recent development of the study of connections between approximation processes and evolution problems, through semigroup theory. In this frame, we refer to the papers of Altomare [1, 2, 4], Felbecker [10, 11] and Campiti [7, 8], where these connections have been successively deepened and the class of evolution equations whose solutions can be approximated by constructive approximation processes has been consistently enlarged. In some cases, the introduction of new types of operators became necessary (see, e.g., [3], [4] and [6]). For a unified treatment of this subject see [5].

Our operators are defined in a quite elementary way, but, at the same time, they apply to approximate the solutions of a wide class of evolution problems. In this paper we restrict ourselves to examine the main properties of the approximation process; connections with semigroup theory will be explored in [9].

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Inspired by the definition of the classical Bernstein polynomials, we replace the binomial coefficients by general ones satisfying similar recursive properties. Actually, we replace the sequences of constant value 1 at the sides of Pascal's triangle with arbitrary ones and define the coefficients of our operators with the same rule of binomial coefficients.

The sequences of linear operators obtained in this way need not to converge in general to the identity operator; indeed, we shall prove the convergence to a multiplication operator by an analytic function depending on the sequences at the sides of Pascal's triangle.

We obtain a decomposition of the classical Bernstein polynomials as sum of elementary operators; our operators will be linear combinations of these last ones.

Qualitative properties and regularity results are stated. Among these, we point out a Voronovskaja-type formula where the second order derivative is perturbed by a first order term depending again on the fixed sequences. This yields the link with semigroup theory and evolution problems (explored in [9]).

We begin to fix some notation. We shall consider polynomial type operators having the form

$$A_n(f)(x) := \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in \mathscr{C}([0,1]),\ x \in [0,1]. \tag{1.1}$$

We shall assume that the coefficients satisfy the following recursive formulas

$$\alpha_{n+1,k} = \alpha_{n,k} + \alpha_{n,k-1}, \qquad k = 1, ..., n \tag{1.2}$$

$$\alpha_{n,0} = \lambda_n, \qquad \alpha_{n,n} = \rho_n, \tag{1.3}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ are fixed sequences of real numbers.

Obviously, if $\lambda_m = \rho_m = 1$ for every $m = 1, ..., n$, we have $\alpha_{n,k} = \binom{n}{k}$ for every $k = 0, ..., n$. Hence, in this case the operator A_n coincides with the classical nth *Bernstein operator* $B_n : \mathscr{C}([0,1] \to \mathscr{C}([0,1])$ defined by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad f \in \mathscr{C}([0,1]). \tag{1.4}$$

It is easy to recognize that condition $A_n(\mathbf{1}) = \mathbf{1}$ is satisfied for every $n \geqslant 1$ only by Bernstein operators; nevertheless, if we require $A_n(\mathbf{1}) = \mathbf{1}$ only for a fixed integer $n \geqslant 1$ we have more possibilities.

We observe that the coefficients and consequently the operators A_n are determined uniquely by the two sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ and $\rho = (\rho_n)_{n\in\mathbb{N}}$. If necessary, we shall write $A_{n,\lambda,\rho}$ to indicate the operator A_n corresponding to the sequences λ and ρ.

Remarks. 1. It is easy to see, using (1.1)–(1.3), that $A_{n,\lambda,\rho}$ depends linearly on the sequences $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ and $\rho=(\rho_n)_{n\in\mathbb{N}}$.

2. We also observe that if the sequences $\xi=(\xi_n)_{n\in\mathbb{N}}$, $\eta=(\eta_n)_{n\in\mathbb{N}}$, $\sigma=(\sigma_n)_{n\in\mathbb{N}}$ and $\tau=(\tau_n)_{n\in\mathbb{N}}$ satisfy the following conditions

$$\xi_n\leqslant\eta_n, \qquad \sigma_n\leqslant\tau_n$$

for every $n\geqslant 1$, then we have

$$A_{n,\xi,\sigma}\leqslant A_{n,\eta,\tau} \tag{1.5}$$

for every $n\in\mathbb{N}$ (i.e., $A_{n,\xi,\sigma}(f)\leqslant A_{n,\eta,\tau}(f)$ for every positive $f\in\mathscr{C}([0,1])$).

In particular, if M is an upper bound for the sequences $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ and $\rho=(\rho_n)_{n\in\mathbb{N}}$), i.e., $\lambda_n\leqslant M$ and $\rho_n\leqslant M$, then $A_{n,\lambda,\rho}\leqslant M\cdot B_n$ (see (1.4)).

By the preceding remarks, it will be useful to consider the mth *left* (*right*, respectively) *elementary operators* which are associated to the sequences $\lambda=(\delta_n^m)_{n\in\mathbb{N}}$ and $\rho=0$ ($\lambda=0$ and $\rho=(\delta_n^m)_{n\in\mathbb{N}}$, respectively). We shall denote by $L_{m,n}$ ($R_{m,n}$, respectively) these operators and by $l_{m,n,k}$ ($r_{m,n,k}$, respectively) their coefficients satisfying (1.2) and (1.3). By using the recursive relations (1.2), it is easy to check the following formulas for the coefficients $l_{m,n,k}$ and $r_{m,n,k}$

$$l_{m,n,k}=\begin{cases}0, & \text{if } n<m;\ n=m,\ k\geqslant 1;\ n>m,\ k=0;\\ & \qquad n>m,\ k\geqslant n-m+1;\\ 1, & \text{if } n=m,\ k=0;\\ \dbinom{n-m-1}{k-1}, & \text{if } n>m,\ 1\leqslant k\leqslant n-m;\end{cases} \tag{1.6}$$

and

$$r_{m,n,k}=\begin{cases}0, & \text{if } n<m;\ n=m,\ k\leqslant n-1;\ n>m,\\ & \qquad k\leqslant m-1;\ n>m,\ k=n;\\ 1, & \text{if } n=m,\ k=n;\\ \dbinom{n-m-1}{k-m}, & \text{if } n>m,\ m\leqslant k\leqslant n-1;\end{cases} \tag{1.7}$$

for every $m\geqslant 1$, $n\geqslant 1$ and $k=0, ..., n$. As a consequence, we obtain

$$L_{m,n}(f)(x)=\sum_{k=1}^{n-m}\binom{n-m-1}{k-1}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right), \qquad \text{if } m<n,$$

$$L_{n,n}(f)(x)=(1-x)^n f(0), \tag{1.8}$$

and

$$R_{m,n}(f)(x) = \sum_{k=m}^{n-1} \binom{n-m-1}{k-m} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad \text{if} \quad m < n,$$
$$R_{n,n}(f)(x) = x^n f(1) \tag{1.9}$$

for every $f \in \mathscr{C}([0, 1])$.

By recalling that

$$B_n(\mathbf{1}) = \mathbf{1}, \qquad B_n(id) = id, \qquad B_n(id^2) = \frac{n-1}{n}\, id^2 + \frac{1}{n}\, id \tag{1.10}$$

(where $id(x) = x$ for every $x \in [0, 1]$), we easily obtain

$$L_{m,n}(\mathbf{1})(x) = \begin{cases} x(1-x)^m, & \text{if} \quad n > m \\ (1-x)^n, & \text{if} \quad n = m, \end{cases}$$
$$R_{m,n}(\mathbf{1})(x) = \begin{cases} (1-x)x^m, & \text{if} \quad n > m, \\ x^n, & \text{if} \quad n = m, \end{cases} \tag{1.11}$$

$$L_{m,n}(id)(x) = \begin{cases} x(1-x)^m \left(\dfrac{1}{n} + \dfrac{n-m-1}{n}\, x\right), & \text{if} \quad n > m, \\ 0, & \text{if} \quad n = m, \end{cases}$$
$$R_{m,n}(id)(x) = \begin{cases} (1-x)x^m \left(\dfrac{m}{n} + \dfrac{n-m-1}{n}\, x\right), & \text{if} \quad n > m, \\ x^n, & \text{if} \quad n = m, \end{cases} \tag{1.12}$$

$$L_{m,n}(id^2)(x) = \begin{cases} x(1-x)^m \left(\dfrac{1}{n^2} + 3\, \dfrac{n-m-1}{n^2}\, x + \dfrac{(n-m-1)(n-m-2)}{n^2}\, x^2\right), \\ \qquad \text{if} \quad n > m, \\ 0, \qquad \text{if} \quad n = m, \end{cases} \tag{1.13}$$

$$R_{m,n}(id^2)(x) = \begin{cases} (1-x)x^m \left(\dfrac{m^2}{n^2} + (1+2m)\, \dfrac{n-m-1}{n^2}\, x + \dfrac{(n-m-1)(n-m-2)}{n^2}\, x^2\right), \\ \qquad \text{if} \quad n > m, \\ x^n, \qquad \text{if} \quad n = m. \end{cases}$$

Finally, we point out the following decomposition

$$A_n = \sum_{m=1}^{n} \lambda_m L_{m,n} + \sum_{m=1}^{n} \rho_m R_{m,n}, \tag{1.14}$$

which is a consequence of Remark 1. In particular, $B_n = \sum_{m=1}^{n} L_{m,n} + \sum_{m=1}^{n} R_{m,n}$.

2. GENERAL CONVERGES PROPERTIES

In this section we fix two arbitrary sequences $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ and $\rho = (\rho_n)_{n \in \mathbb{N}}$ of real numbers and investigate the convergence of the sequence $(A_n)_{n \in \mathbb{N}}$.

Since for every $f \in \mathscr{C}([0, 1])$ we have

$$A_n(f)(0) = \lambda_n f(0), \qquad A_n(f)(1) = \rho_n f(1), \tag{2.1}$$

the convergence of $(A_n)_{n \in \mathbb{N}}$ implies that of the sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$.

So, we assume that $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$ converge and put

$$\lambda_\infty := \lim_{n \to \infty} \lambda_n, \qquad \rho_\infty := \lim_{n \to \infty} \rho_n. \tag{2.2}$$

Now, representation (1.14) and formulas (1.11) yield

$$A_n(\mathbf{1})(x) = \sum_{m=1}^{n-1} (\lambda_m x(1-x)^m + \rho_m x^m(1-x)) + \lambda_n(1-x)^n + \rho_n x^n; \tag{2.3}$$

this suggests that we consider the power series

$$\sum_{m=1}^{\infty} \lambda_m (1-x)^m \qquad \text{and} \qquad \sum_{m=1}^{\infty} \rho_m x^m$$

which have radii of convergence greater than or equal to 1 because of the boundedness of $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$. As a consequence, we can define the functions

$$\ell(x) := \begin{cases} \lambda_\infty, & \text{if} \quad x = 0, \\ \displaystyle\sum_{m=1}^{\infty} \lambda_m x(1-x)^m, & \text{if} \quad 0 < x \leqslant 1, \end{cases}$$

$$\imath(x) := \begin{cases} \displaystyle\sum_{m=1}^{\infty} \rho_m x^m (1-x), & \text{if} \quad 0 \leqslant x < 1, \\ \rho_\infty, & \text{if} \quad x = 1, \end{cases} \tag{2.4}$$

which turn out to be continuous on $[0,1]$ by the convergence of $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$.
Observe that $|\ell(x)|\leqslant(1-x)\cdot\sup_{m\geqslant 1}|\lambda_m|$ and $|r(x)|\leqslant x\cdot\sup_{m\geqslant 1}|\rho_m|$. Moreover, putting $a_m=\lambda_m-\lambda_{m-1}$ and $b_m=\rho_m-\rho_{m-1}$ for every $m\geqslant 1$ with the convention $\lambda_0=\rho_0=0$, we have

$$\lambda_m=\sum_{k=1}^{m} a_k, \qquad \rho_m=\sum_{k=1}^{m} b_k$$

and $\ell(x)=\sum_{m=1}^{\infty} a_m(1-x)^m$, $r(x)=\sum_{m=1}^{\infty} b_m x^m$.

The function

$$w:=\ell+r \tag{2.5}$$

plays a central role in the study of the sequence $(A_n)_{n\in\mathbb{N}}$; in order to estimate its degree of convergence, we observe that the limit operator depends also on the asymptotic behavior of the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$. As a consequence, our estimate will involve the modulus of continuity

$$\omega(f,\delta):=\sup\{|f(x)-f(y)|\mid x,y\in[0,1],\ |x-y|\leqslant\delta\}, \tag{2.6}$$

and also a "remainder term"

$$r(n):=\sup_{m\geqslant n}\max\{|\lambda_m-\lambda_n|,\ |\rho_m-\rho_n|\}. \tag{2.7}$$

We define, for simplicity

$$s(n):=\max_{m\leqslant n}\{|\lambda_m|,\ |\rho_m|\}.$$

THEOREM 2.1. *For every $f\in\mathscr{C}([0,1])$ and $x\in[0,1]$*

$$|A_n(f)(x)-f(x)\,A_n(\mathbf{1})(x)|\leqslant(1+x(1-x))\ s(n)\ \omega\left(f,\frac{1}{\sqrt{n}}\right) \tag{2.9}$$

and therefore

$$|A_n(f)(x)-w(x)\cdot f(x)|\leqslant(1+x(1-x))\ s(n)\ \omega\left(f,\frac{1}{\sqrt{n}}\right) + ((1-x)^n+x^n)\ r(n)\ |f(x)|. \tag{2.10}$$

In particular, the following uniform estimate holds

$$\|A_n(f)-w\cdot f\| \leqslant \frac{5}{4}\, s(n)\, \omega\left(f, \frac{1}{\sqrt{n}}\right)+r(n)\, \|f\|. \tag{2.11}$$

Proof. First, we observe that, for every $f\in\mathscr{C}([0,1])$ and $x\in[0,1]$, by (2.8) and Remark 2 in Section 1, we have

$$|A_n(f)(x)| \leqslant \sum_{k=0}^{n} |\alpha_{n,k}|\, x^k(1-x)^{n-k} \left|f\left(\frac{k}{n}\right)\right| \leqslant s(n)\, B_n(|f|)(x).$$

For every $\delta>0$, by using the well-known inequality $|f(y)-f(x)|\leqslant (1+(1/\delta^2)(x-y)^2)\,\omega(f,\delta)$, we obtain

$$\begin{aligned}
&|A_n(f)(x)-f(x)\, A_n(\mathbf{1})(x)| \\
&\quad\leqslant \sum_{k=0}^{n} |\alpha_{n,k}|\, x^k(1-x)^{n-k} \left|f\left(\frac{k}{n}\right)-f(x)\right| \\
&\quad\leqslant s(n)\, \omega(f,\delta) \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}\left(1+\frac{1}{\delta^2}\left(x-\frac{k}{n}\right)^2\right) \\
&\quad=\left(1+\frac{1}{\delta^2}\,\frac{x(1-x)}{n}\right) s(n)\, \omega(f,\delta).
\end{aligned}$$

Therefore, by taking $\delta=1/\sqrt{n}$, (2.9) immediately follows.

On the other hand, by (2.3) and (2.4), for every $x\in\,]0,1[$,

$$\begin{aligned}
&|A_n(\mathbf{1})(x)-w(x)| \\
&\quad= \left|\lambda_n(1-x)^n+\rho_n x^n-\sum_{m=n}^{\infty}\lambda_m x(1-x)^m-\sum_{m=n}^{\infty}\rho_m x^m(1-x)\right| \\
&\quad=\left|(1-x)^n\sum_{m=0}^{\infty}(\lambda_n-\lambda_{n+m})\, x(1-x)^m+x^n\sum_{m=0}^{\infty}(\rho_n-\rho_{n+m})\, x^m(1-x)\right| \\
&\quad\leqslant ((1-x)^n+x^n)\, r(n)
\end{aligned} \tag{1}$$

and the same inequality obviously holds if $x=0$ or $x=1$.

Hence, we have obtained

$$\begin{aligned}
&|A_n(f)(x)-w(x)\, f(x)| \\
&\quad\leqslant |A_n(f)(x)-f(x)\, A_n(\mathbf{1})(x)|+|f(x)\, A_n(\mathbf{1})(x)-w(x)\, f(x)| \\
&\quad\leqslant (1+x(1-x))\, s(n)\, \omega\left(f,\frac{1}{\sqrt{n}}\right)+((1-x)^n+x^n)\, r(n)\, |f(x)|,
\end{aligned}$$

and this completes the proof. ∎

By the preliminary remarks and Theorem 2.1, we can completely describe the convergence of $(A_n)_{n\in\mathbb{N}}$.

THEOREM 2.2. *The sequence $(A_n)_{n\in\mathbb{N}}$ converges strongly on $\mathscr{C}([0,1])$ if and only if the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge.*

In this case, if w denotes the function defined by (2.5), *we have*

$$\lim_{n\to\infty} A_n(f) = w\cdot f \qquad \text{uniformly on } [0,1] \tag{2.12}$$

for every $f\in\mathscr{C}([0,1])$.

Now we consider some concrete examples.

EXAMPLES. 1. In the case of Bernstein operators, we have $r(n)=0$ and $s(n)=1$ for every $n\geqslant 1$ and therefore (2.9) reduces to the well-known result of Lorentz [12, p. 20]:

$$|B_n(f)(x)-f(x)|\leqslant (1+x(1-x))\,\omega(f, n^{-1/2}). \tag{2.13}$$

2. We can decompose the classical n-th Bernstein operator into the sum of its left and right part. Namely, for every $n\geqslant 1$, we define the nth *left Bernstein operator* B_n^l corresponding to the sequences $\lambda_k=1$ and $\rho_k=0$ for every $k\geqslant 1$ and similarly, the nth *right Bernstein operator* B_n^r by considering the sequences $\lambda_k=0$ and $\rho_k=1$ for every $k\geqslant 1$.

It is easy to show that

$$B_n^l(f)(x)=\sum_{k=0}^{n-1}\binom{n-1}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{2.14}$$

and

$$B_n^r(f)(x)=\sum_{k=1}^{n}\binom{n-1}{k-1} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{2.15}$$

In this case $\ell(x)=1-x$ and $\imath(x)=0$ for the left Bernstein operators and $\ell(x)=0$ and $\imath(x)=x$ for the right Bernstein operators.

Hence, by Theorem 2.2, we get, for every $f\in\mathscr{C}([0,1])$

$$\lim_{n\to\infty} B_n^l(f)(x)=(1-x)\,f(x) \text{ uniformly in } x\in[0,1] \tag{2.16}$$

and

$$\lim_{n \to \infty} B_n^r(f)(x) = xf(x) \qquad \text{uniformly in} \quad x \in [0, 1]. \tag{2.17}$$

Finally, Theorem 2.1 yields the following quantitative estimates:

$$\|B_n^l(f) - (1 - id) f\| \leqslant \frac{5}{4}\, \omega\left(f, \frac{1}{\sqrt{n}}\right), \tag{2.18}$$

and

$$\|B_n^r(f) - id\, f\| \leqslant \frac{5}{4}\, \omega\left(f, \frac{1}{\sqrt{n}}\right). \tag{2.19}$$

3. For every $n \geqslant 1$ and $m \geqslant 1$, we can also define the *m-truncation* $B_{m,n}$ *of the nth Bernstein operator* as follows

$$B_{m,n} := \sum_{j=1}^{m} L_{j,n} + R_{j,n}. \tag{2.20}$$

By (1.8) and (1.9) it follows $B_{m,n} = B_n$ if $m \geqslant n$ while, if $m < n$,

$$\begin{aligned} B_{m,n}(f)(x) = \sum_{j=1}^{m} \Bigg(\sum_{k=1}^{n-j} \binom{n-j-1}{k-1} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \\ + \sum_{k=j}^{n-1} \binom{n-j-1}{k-j} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)\Bigg). \end{aligned}$$

Since

$$\ell(x) = (1-x) - (1-x)^{m+1}, \qquad \imath(x) = x - x^{m+1},$$

we have

$$\begin{aligned} &\lim_{n \to \infty} B_{m,n}(f)(x) \\ &\quad = (1 - (1-x)^{m+1} - x^{m+1})\, f(x) \qquad \text{uniformly in} \quad x \in [0, 1] \end{aligned} \tag{2.21}$$

and

$$\|B_{m,n}(f) - (1 - (1 - id)^{m+1} - id^{m+1}) \cdot f\| \leqslant \frac{5}{4}\, \omega\left(f, \frac{1}{\sqrt{n}}\right). \tag{2.22}$$

4. A situation of particular interest can be obtained by considering linear combinations of Bernstein operators and a finite number of elementary operators. This is equivalent to assume that the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ are definitively constant. If $\lambda_n=\lambda$ and $\rho_n=\rho$ for every $n\geqslant p$, we have (see (2.4) and (2.5))

$$w(x)=\lambda(1-x)+\rho x+\sum_{m=1}^{p-1}((\lambda_m-\lambda)\,x(1-x)^m+(\rho_m-\rho)(1-x)x^m) \tag{2.23}$$

and hence w is a polynomial of degree at most p. Obviously, for the associated sequence $(A_n)_{n\in\mathbb{N}}$ defined by (1.1), for every $n\geqslant p$ we have $r(n)=0$ and consequently by (2.11)

$$\|A_n(f)-w\cdot f\|\leqslant\frac{5}{4}\max_{m\leqslant p}\{|\lambda_m|,|\rho_m|\}\;\omega\left(f,\frac{1}{\sqrt{n}}\right). \tag{2.24}$$

Conversely, observe that every polynomial of degree at most p can be written as in (2.23) from which we can obtain the corresponding definitively constant sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$.

As regards to the sequences of elementary operators $(L_{m,n})_{n\in\mathbb{N}}$ and $(R_{m,n})_{n\in\mathbb{N}}$ defined by (1.8) and (1.9), we explicitly observe that

$$\lim_{n\to\infty} L_{m,n}(f)(x)=x(1-x)^m f(x) \qquad \text{uniformly in} \quad x\in[0,1] \tag{2.25}$$

and

$$\lim_{n\to\infty} R_{m,n}(f)(x)=x^m(1-x)\,f(x) \qquad \text{uniformly in} \quad x\in[0,1] \tag{2.26}$$

for every $f\in\mathscr{C}([0,1])$. Moreover, we can state the following more precise quantitative estimates.

Proposition 2.3. *For every $m=1,...,n-2$ and $f\in\mathscr{C}([0,1])$, we have*

$$\|L_{m,n}(f)-\varphi\cdot f\|\leqslant\begin{cases}\dfrac{9}{8}\,\dfrac{1}{m+1}\,\omega\left(f,\dfrac{1}{\sqrt{n-m-1}}\right), & \text{if} \quad m\leqslant\sqrt{n-3/4}-\dfrac{1}{2},\\[2ex] \dfrac{9}{8}\,\dfrac{1}{m+1}\,\omega\left(f,\dfrac{m}{n-m-1}\right), & \text{if} \quad m\geqslant\sqrt{n-3/4}-\dfrac{1}{2},\end{cases} \tag{2.27}$$

where $\varphi(x) := x(1-x)^m$ *and*

$$\|R_{m,n}(f) - \psi \cdot f\| \leqslant \begin{cases} \dfrac{9}{8}\,\dfrac{1}{m+1}\,\omega\left(f, \dfrac{1}{\sqrt{n-m-1}}\right), & \text{if } m \leqslant \sqrt{n-3/4} - \dfrac{1}{2}, \\ \dfrac{9}{8}\,\dfrac{1}{m+1}\,\omega\left(f, \dfrac{m}{n-m-1}\right), & \text{if } m \geqslant \sqrt{n-3/4} - \dfrac{1}{2}, \end{cases} \tag{2.28}$$

where $\psi(x) := x^m(1-x)$.

Proof. By (1.8) and (2.13), we obtain, for every $f \in \mathscr{C}([0,1])$,

$$\begin{aligned} &|L_{m,n}(f)(x) - x(1-x)^m f(x)| \\ &\quad \leqslant x(1-x)^m \Bigg(|B_{n-m-1}(f)(x) - f(x)| \\ &\qquad + \sum_{h=0}^{n-m-1} \binom{n-m-1}{h} x^h (1-x)^{n-m-1-h} \left| f\left(\frac{h+1}{n}\right) - f\left(\frac{h}{n-m-1}\right) \right| \Bigg) \\ &\quad \leqslant x(1-x)^m \left((1 + x(1-x))\,\omega\left(f, \frac{1}{\sqrt{n-m-1}}\right) + \omega\left(f, \frac{n}{n-m-1}\right) \right). \end{aligned}$$

At this point, we observe that $1/\sqrt{n-m-1} \leqslant m/(n-m-1)$ exactly when $m \geqslant \sqrt{n-3/4} - \frac{1}{2}$. Hence, by the above inequalities,

$$\begin{aligned} &|L_{m,n}(f)(x) - x(1-x)^m f(x)| \\ &\quad \leqslant \begin{cases} x(1-x)^m\,(2 + x(1-x))\,\omega\left(f, \dfrac{1}{\sqrt{n-m-1}}\right), & \text{if } m \leqslant \sqrt{n-3/4} - \frac{1}{2}, \\ x(1-x)^m\,(2 + x(1-x))\,\omega\left(f, \dfrac{m}{n-m-1}\right), & \text{if } m \geqslant \sqrt{n-3/4} - \frac{1}{2}, \end{cases} \end{aligned}$$

and this yields (2.27) since $x(1-x)^m \leqslant 1/(2(m+1))$ when $x \in [0,1]$.

The proof of (2.28) is analogous. ∎

We conclude this section by examining the connection between $w(x)$ and the coefficients $\alpha_{n,k}$, with k/n around x. For, we write

$$A_n(f)(x) = \sum_{k=0}^{n} b_{n,k} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{2.29}$$

where

$$b_{n,k} := \alpha_{n,k} \binom{n}{k}^{-1}, \qquad k = 0, ..., n. \tag{2.30}$$

By representation (1.14) it follows

$$b_{n,k} = \sum_{m=1}^{n} (\lambda_m b^l_{m,n,k} + \rho_m b^r_{m,n,k}), \tag{2.31}$$

where $b^l_{m,n,k} := l_{m,n,k} \binom{n}{k}^{-1}$ and $b^r_{m,n,k} := r_{m,n,k} \binom{n}{k}^{-1}$. By using formulas (1.6) and (1.7), we obtain

$$b^l_{m,n,k} = \begin{cases} \dfrac{k}{n} \displaystyle\prod_{j=1}^{m} \dfrac{1-(k+j-1)/n}{1-j/n}, & m < n, \\ 1, & m = n,\ k = 0, \\ 0, & m > n \text{ or } (m = n \text{ and } k > 0), \end{cases} \tag{2.32}$$

$$b^r_{m,n,k} = \begin{cases} \left(1 - \dfrac{k}{n}\right) \displaystyle\prod_{j=1}^{m} \dfrac{k-j+1}{n-j}, & m < n, \\ 1, & m = n,\ k = n, \\ 0, & m > n \text{ or } (m = n \text{ and } k < n). \end{cases} \tag{2.33}$$

It is also useful to consider the continuous piecewise affine function b_n satisfying the conditions

$$b_n\left(\frac{k}{n}\right) = b_{n,k}, \qquad k = 0, ..., n. \tag{2.34}$$

So we can write $A_n(\mathbf{1}) = B_n(b_n)$ for every $n \in \mathbb{N}$. Moreover, comparing (2.30) with (1.2) and (1.3), we obtain

$$b_n(0) = \lambda_n, \qquad b_n(1) = \rho_n,$$

$$b_{n+1}\left(\frac{k}{n+1}\right) = \left(1 - \frac{k}{n+1}\right) b_n\left(\frac{k}{n}\right) + \frac{k}{n+1}\, b_n\left(\frac{k-1}{n}\right), \quad k = 1, ..., n. \tag{2.35}$$

THEOREM 2.4. *The sequence $(b_n)_{n \in \mathbb{N}}$ converges uniformly to w.*

Proof. As a first step, we show that $\lim_{n \to \infty,\, k/n \to x} b_{n,k} = w(x)$ uniformly in $x \in [0, 1]$, that is, for every $\varepsilon > 0$ there exist $\delta > 0$ and $\nu \in \mathbb{N}$

such that $|b_{n,k} - w(x)| < \varepsilon$ whenever $n > \nu$, $|k/n - x| < \delta$, Indeed, formulas (2.32) and (2.33) imply

$$\lim_{\substack{n \to \infty \\ k/n \to x}} b^l_{m,n,k} = x(1-x)^m \qquad \text{uniformly in} \quad x \in [0, 1] \tag{1}$$

and

$$\lim_{\substack{n \to \infty \\ k/n \to x}} b^r_{m,n,k} = x^m(1-x) \qquad \text{uniformly in} \quad x \in [0, 1]. \tag{2}$$

We write

$$A_n = \sum_{m=1}^{n} ((\lambda_m - \lambda_\infty) L_{m,n} + (\rho_m - \rho_\infty) R_{m,n}) + \lambda_\infty B^l_n + \rho_\infty B^r_n. \tag{3}$$

If we denote $b^l_{n,k}$ and $b^r_{n,k}$ the coefficients of B^l_n and B^r_n, respectively, a direct computation shows that

$$b^l_{n,k} = 1 - \frac{k}{n} \qquad \text{and} \qquad b^r_{n,k} = \frac{k}{n}, \qquad k = 0, ..., n. \tag{4}$$

Moreover, if we put $\lambda_m = \rho_m = 1$ in (2.31), we obtain

$$\sum_{m=1}^{n} (b^l_{m,n,k} + b^r_{m,n,k}) = 1. \tag{5}$$

By (3) and (4) we can restrict ourselves to the case $\lambda_\infty = \rho_\infty = 0$. For every $\nu < n$, we have

$$\begin{aligned} |b_{n,k} - w(x)| \leqslant & \left| \sum_{m=1}^{\nu} (\lambda_m b^l_{m,n,k} + \rho_m b^r_{m,n,k}) \right. \\ & \left. - \sum_{m=1}^{\nu} (\lambda_m x(1-x)^m + \rho_m x^m(1-x)) \right| \\ & + \left| \sum_{m=\nu+1}^{n} (\lambda_m b^l_{m,n,k} + \rho_m b^r_{m,n,k}) \right| \\ & + \left| \sum_{m=\nu+1}^{\infty} (\lambda_m x(1-x)^m + \rho_m x^m(1-x)) \right|. \end{aligned}$$

The second and the third term in the last inequality can be estimated with $\sup_{m \geqslant \nu} \{|\lambda_n|, |\rho_n|\}$ (use (5) for the second term), and they converge to 0 if $\nu \to \infty$. The first term converges to 0, ν being fixed, uniformly in $x \in [0, 1]$ if $n \to \infty$ and $k/n \to x$ by (1) and (2).

At this point, we can show the uniform convergence of the sequence $(b_n)_{n \in \mathbb{N}}$. If $\varepsilon > 0$, by the first part, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $|b_{n,k} - w(x)| < \varepsilon$ whenever $n > \nu$, $|k/n - x| < \delta$. We can assume $1/\delta < \nu$ and $|w(x) - w(y)| < \varepsilon$ whenever $x, y \in [0, 1]$, $|x - y| < \delta$. Let $n > \nu$; if $x \in [0, 1]$, there exists $k = 0, ..., n-1$ such that $k/n \leqslant x \leqslant (k+1)/n$; moreover, we have

$$b_n(x) = t b_n\left(\frac{k}{n}\right) + (1-t) b_n\left(\frac{k+1}{n}\right)$$

for some $0 \leqslant t \leqslant 1$. Hence

$$|b_n(x) - w(x)| \leqslant t \left| b_n\left(\frac{k}{n}\right) - w(x) \right| + (1-t) \left| b_n\left(\frac{k+1}{n}\right) - w(x) \right| \leqslant \varepsilon$$

and this completes the proof. ∎

The preceding result allows us to derive some qualitative properties of the function w by studying the sequence $(b_n)_{n \in \mathbb{N}}$.

PROPOSITION 2.5. *The following properties hold*:

(1) *If* $(\lambda_n)_{n \in \mathbb{N}}$ *and* $(\rho_n)_{n \in \mathbb{N}}$ *are both increasing, then* $(b_n)_{n \in \mathbb{N}}$ *is an increasing sequence of convex functions and* w *is convex.*

(2) *If* $(\lambda_n)_{n \in \mathbb{N}}$ *and* $(\rho_n)_{n \in \mathbb{N}}$ *are both decreasing, then* $(b_n)_{n \in \mathbb{N}}$ *is a decreasing sequence of concave functions and* w *is concave.*

(3) *If* $(\lambda_n)_{n \in \mathbb{N}}$ *is increasing and* $(\rho_n)_{n \in \mathbb{N}}$ *is decreasing and if* $\lambda_1 \geqslant \rho_1$, *then* w *and every* b_n *are decreasing.*

(4) *If* $(\lambda_n)_{n \in \mathbb{N}}$ *is decreasing and* $(\rho_n)_{n \in \mathbb{N}}$ *is increasing and if* $\lambda_1 \leqslant \rho_1$, *then* w *and every* b_n *are decreasing.*

(5) *If* $\lambda_n = \rho_n$ *for every* $n \geqslant 1$, *then, for every* $n \geqslant 1$ *and* $x \in [0, 1]$, $b_n(1-x) = b_n(x)$ *and consequently* $w(1-x) = w(x)$.

Proof. Let $n \geqslant 1$ and, for every $i = 1, ..., n$, denote by $r_{n,i}$ the segment of b_n joining the points $((i-1)/n, b_{n,i-1})$ and $(i/n, b_{n,i})$, and by $m_{n,i} = n(b_{n,i} - b_{n,i-1})$ its angular coefficient.

We observe that b_n is convex (respectively, concave) if and only if $m_{n,1} \leqslant \cdots \leqslant m_{n,n}$ (respectively, $m_{n,1} \geqslant \cdots \geqslant m_{n,n}$) and moreover b_n is increasing (respectively, decreasing) if and only if all $m_{n,i}$, $i = 1, ..., n$, are positive (respectively, negative).

We also point out that, for every $n \geqslant 1$ and $i = 2, ..., n$, the value of $m_{n+1,i}$ is always between the values $m_{n,i-1}$ and $m_{n,i}$ since, by (2.35), the endpoints of $r_{n+1,i}$ are interior points of $r_{n,i-1}$ and $r_{n,i}$.

At this point, the proof proceeds by induction on $n \geqslant 1$.

Under the assumption (1), if b_n is convex, by the inequalities $\lambda_n \leqslant \lambda_{n+1}$ and $\rho_n \leqslant \rho_{n+1}$, and by the above argument, we obtain

$$m_{n+1,1} \leqslant m_{n,1} \leqslant m_{n+1,2} \leqslant m_{n,2} \leqslant \cdots \leqslant m_{n+1,n} \leqslant m_{n,n} \leqslant m_{n+1,n+1};$$

therefore the function b_{n+1} is also convex. Moreover, $b_n \leqslant b_{n+1}$ since each affine restriction of b_{n+1} is a secant of the convex function b_n.

The proof of property (2) is analogous.

The sequence $(\lambda_n)_{n\in\mathbb{N}}$ is decreasing if and only if $(m_{n,1})_{n\in\mathbb{N}}$ is increasing and similarly $(\rho_n)_{n\in\mathbb{N}}$ is increasing if and only if $(m_{n,n})_{n\in\mathbb{N}}$ is increasing for every $n \geqslant 1$.

Since $m_{n+1,i}$ is a value between $m_{n,i-1}$ and $m_{n,i}$ for every $i = 2, ..., n$, property (3) will follow by induction on $n \geqslant 1$ provided that $m_{1,1} \geqslant 0$, i.e., $\lambda_1 \geqslant \rho_1$. The proof of property (4) is similar.

Finally, if $\lambda_n = \rho_n$ for every $n \geqslant 1$, then $\alpha_{n,n-k} = \alpha_{n,k}$ for every $k = 0, ..., n$ and consequently, by (2.30), $b_{n,n-k} = b_{n,k}$. Hence, property (5) follows by the definition of b_n. ∎

Remark. Observe that in general there is no continuous function g satisfying $g(k/n) = b_{n,k}$ for all $n \in \mathbb{N}$ and $k = 0, ..., n$. In fact, in this case we should have $b_n = g$ for every $n \geqslant 1$ and therefore $\lambda_n = g(0)$ and $\rho_n = g(1)$ for every $n \geqslant 1$. It follows that the only possibility is that the function $w(x) = g(0)(1-x) + g(1)x$ is affine on $[0,1]$.

3. REGULARITY RESULTS

Our purpose is to improve estimates (2.10) and (2.11) when f satisfies suitable regularity properties. We shall prove also a Voronovskaja-type formula for the operators A_n.

We keep the same notation of the preceding section. We assume that the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge and define

$$\|\lambda\| := \sup_{n\in\mathbb{N}} |\lambda_n|, \qquad \|\rho\| := \sup_{n\in\mathbb{N}} |\rho_n| \qquad \text{and} \qquad M := \max\{\|\lambda\|, \|\rho\|\}.$$

The following two lemmas play a central role throughout this section.

LEMMA 3.1. *The following properties hold*:

(i) $\|id \cdot A_n(\mathbf{1}) - A_n(id)\| \leqslant \dfrac{3M}{n}$.

(ii) $$\lim_{n\to\infty} n(A_n(id)(x) - xA_n(\mathbf{1})(x)) = \begin{cases} x(1-x)\, w'(x), & \textit{if } 0<x<1, \\ 0, & \textit{if } \ x=0 \textit{ or } x=1, \end{cases}$$

uniformly on $[0, 1]$.

Proof. Using the representation (1.14) and formulas (1.11), (1.12) and (1.13), we directly obtain

$$\begin{aligned} x \cdot A_n(\mathbf{1})(x) - A_n(id)(x) &= \sum_{m=1}^{n-1} \left(\lambda_m x(1-x)^m \left(\frac{m+1}{n}\, x - \frac{1}{n} \right) \right. \\ &\quad \left. + \rho_m x^m (1-x) \left(\frac{m+1}{n}\,(x-1) + \frac{1}{n} \right) \right) \\ &\quad + \lambda_n x(1-x)^n + \rho_n x^n (1-x). \end{aligned} \tag{1}$$

First we prove that, for $0<x<1$,

$$\begin{aligned} x(1-x)\, w'(x) = -\sum_{m=1}^{\infty} (\lambda_m x(1-x)^m\, ((m+1)x-1) \\ + \rho_m x^m (1-x)((m+1)(x-1)+1)) \end{aligned} \tag{2}$$

and that the second member can be continuously extended at the endpoints.

For, we put $a(x) := \sum_{m=1}^{\infty} \lambda_m x^2 (m+1)(1-x)^m$ and observe that $\lim_{x\to 0} a(x) = \lambda_\infty$. In fact

$$\begin{aligned} |a(x) - \lambda_\infty(1-x^2)| &= \left| \sum_{m=1}^{\infty} (\lambda_m - \lambda_\infty)(m+1)\, x^2 (1-x)^m \right| \\ &\leqslant \sum_{m=1}^{v} |\lambda_m - \lambda_\infty|\, (m+1)\, x^2(1-x)^m \\ &\quad + \sum_{m=v+1}^{\infty} |\lambda_m - \lambda_\infty|\, (m+1)\, x^2 (1-x)^m \end{aligned}$$

for all $v \in \mathbb{N}$, where the second term is less than $\sup_{m\geqslant v} |\lambda_m - \lambda_\infty|$ and the first term, with a fixed v, converges to 0 as $x \to 0$.

Moreover

$$a(x) = \sum_{m=1}^{\infty} \lambda_m x^2 \left(-\frac{d}{dx} \right) (1-x)^{m+1}$$

$$= \sum_{m=1}^{\infty} \lambda_m \left(\left(-\frac{d}{dx} \right) (x^2(1-x)^{m+1}) + 2x(1-x)^{m+1} \right)$$

$$= \left(-\frac{d}{dx} \right) (x(1-x)\, \ell(x)) + 2(1-x)\, \ell(x) = \ell(x) - x(1-x)\, \ell'(x)$$

so that $\lim_{x \to 0} x(1-x)\, \ell'(x) = 0$.

In the same way, defining $b(x) := \sum_{m=1}^{\infty} \rho_m (1-x)^2\, (m+1)\, x^m$, we obtain $b(x) = \mathscr{r}(x) + x(1-x)\ )\mathscr{r}'(x)$, $\lim_{x \to 1} x(1-x)\, \mathscr{r}'(x) = 0$, whence (2) is proved and $\lim_{x \to 0,\, 1} x(1-x)\, w'(x) = 0$.

Now, we note that

$$\left| \sum_{m=n}^{\infty} \lambda_m x^2 (m+1)(1-x)^m \right| \leqslant \sup_{m \geqslant n} |\lambda_m|\, x^2 \sum_{m=1}^{\infty} (m+1)(1-x)^m$$

$$= \sup_{m \geqslant n} |\lambda_m|\, x^2 \left| \frac{d}{dx} \sum_{m=1}^{\infty} (1-x)^{m+1} \right|$$

$$= \sup_{m \geqslant n} |\lambda_m|\, (1-x^2) \leqslant M(1-x^2) \tag{3}$$

and similarly

$$\left| \sum_{m=n}^{\infty} \rho_m (1-x)^2\, (m+1)\, x^m \right| \leqslant \sup_{m \geqslant n} |\rho_m| x^2 \leqslant M x^2. \tag{4}$$

Since

$$\sum_{m=1}^{n-1} (|\lambda_m|\, x(1-x)^m + |\rho_m|\, x^m(1-x)) \leqslant M,$$

$$\sup_{0 \leqslant x \leqslant 1} |x(1-x)^n + x^n(1-x)| \leqslant \frac{1}{n} \tag{5}$$

and, by (1),

$$|x \cdot A_n(\mathbf{1})(x) - A_n(id)(x)|$$

$$\leqslant \sum_{m=1}^{n-1} \left(|\lambda_m|\, x(1-x)^m\, \frac{m+1}{n}\, x + |\rho_m|\, x^m(1-x)\, \frac{m+1}{n}\, (1-x) \right)$$

$$+ \frac{1}{n} \sum_{m=1}^{n-1} (|\lambda_m|\, x(1-x)^m + |\rho_m|\, x^m(1-x))$$

$$+ |\lambda_n|\, x(1-x)^n + |\rho_n|\, x^n(1-x),$$

(i) easily follows from (3), (4) and (5).

To prove (ii) we first assume that $\lambda_\infty = \rho_\infty = 0$. Then, for $0 < x < 1$, by (1) and (2)

$$n(A_n(id)(x) - xA_n(\mathbf{1})(x)) - x(1-x)\, w'(x)$$

$$= \sum_{m=n}^{\infty} (\lambda_m x(1-x)^m ((m+1)x - 1) + \rho_m x^m(1-x)((m+1)(x-1)+1))$$

$$- n\lambda_n x(1-x)^n - n\rho_n x^n(1-x).$$

Using (3), (4) and (5) and the continuity of the function $x(1-x)\, w'(x)$ in $[0,1]$ one easily obtains

$$\sup_{0 \leqslant x \leqslant 1} |n(A_n(id)(x) - xA_n(\mathbf{1})(x)) - x(1-x)\, w'(x)| \leqslant 3 \sup_{m \geqslant n} \{|\lambda_m|, |\rho_m|\}$$

which yields the uniform convergence.

The general case reduces to the previous one by writing

$$A_n = \sum_{m=1}^{n} (\lambda_m - \lambda_\infty) L_{m,n} + \sum_{m=1}^{n} (\rho_m - \rho_\infty) R_{m,n} + \lambda_\infty B_n^l + \rho_\infty B_n^r \qquad (6)$$

and noticing that (ii) immediately follows by direct computation for the operators B_n^l and B_n^r. ∎

LEMMA 3.2. *The equality*

$$\lim_{n\to\infty} nA_n((id - x\cdot\mathbf{1})^2)(x) = x(1-x)\, w(x)$$

holds uniformly in $x \in [0, 1]$.

Proof. We write $A_n = C_n + D_n$, where $C_n := \sum_{m=1}^{n} \lambda_m L_{m,n}$ and $D_n := \sum_{m=1}^{n} \rho_m R_{m,n}$. Using again formulas (1.11), (1.12) and (1.13), we obtain

$$C_n((id - x\cdot\mathbf{1})^2)(x) = \frac{1}{n^2} \sum_{m=1}^{n-1} \lambda_m x(1-x)^m ((m^2+3m-n+2)x^2$$

$$+ (n-3m-3)x + 1) + \lambda_n x^2(1-x)^n.$$

For each $k \in \mathbb{N}$ the power series $\sum_{m=1}^{\infty} m^k x^k (1-x)^m$ are bounded in $[0, 1]$; hence, if $\lambda_\infty = 0$, an argument similar to that of the proof of Lemma 3.1 yields

$$\lim_{n\to\infty} nC_n((id - x\cdot\mathbf{1})^2)(x) = x(1-x)\, \ell(x)$$

uniformly in $x \in [0, 1]$. Since the statement is obviously true for the operator B_n^l, the general case $\lambda_\infty \in \mathbb{R}$ can be obtained by using (6) as in the proof of Lemma 3.1.

In the same way, $\lim_{n\to\infty} nD_n((id - x\cdot\mathbf{1})^2)(x) = x(1-x)\,\imath(x)$ uniformly in $x \in [0, 1]$, and this completes the proof. ∎

PROPOSITION 3.3. *If $f \in \mathscr{C}^1([0, 1])$, then*

$$\|A_n(f) - A_n(\mathbf{1})\cdot f\| \leqslant 3M\left(\frac{\|f'\|}{n} + \frac{1}{4\sqrt{n}}\,\omega\left(f', \frac{1}{\sqrt{n}}\right)\right). \tag{3.1}$$

Proof. Using Lagrange's theorem

$$f\left(\frac{k}{n}\right) - f(x) = \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)(f'(\xi) - f'(x)),$$

we get, for all $\delta > 0$,

$$\begin{aligned}
&|A_n(f)(x) - f(x)\cdot A_n(\mathbf{1})(x)|\\
&\quad\leqslant \left|\sum_{k=0}^{n} \alpha_{n,k}\, x^k(1-x)^{n-k}\left(\frac{k}{n} - x\right) f'(x)\right|\\
&\qquad + M\omega(f', \delta)\sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k}\left|\frac{k}{n} - x\right|\left(1 + \frac{1}{\delta}\left|\frac{k}{n} - x\right|\right);
\end{aligned}$$

choosing $\delta = 1/\sqrt{n}$, the second term in the preceding sum can be estimated with $3M/(4\sqrt{n})\,\omega(f', 1/\sqrt{n})$ in a straightforward way (see, e.g., [12, p. 21]). As regard the first term we have $|A_n(id)(x) - x\cdot A_n(\mathbf{1})(x)|\ |f'(x)| \leqslant (3M/n)\,\|f'\|$ by Lemma 3.1(i). ∎

Remark. If $f \in \mathscr{C}^1([0, 1])$, we have, for every $x \in [0, 1]$,

$$\begin{aligned}
&|A_n(f)(x) - w(x)\cdot f(x)|\\
&\qquad \leqslant |A_n(f)(x) - f(x)\cdot A_n(\mathbf{1})(x)| + |f(x)|\ |A_n(\mathbf{1})(x) - w(x)|
\end{aligned} \tag{3.2}$$

and therefore, by (3.1) and (1) in the proof of Theorem 2.1,

$$\|A_n(f) - w\cdot f\| \leqslant 3M\left(\frac{\|f'\|}{n} + \frac{1}{4\sqrt{n}}\,\omega\left(f', \frac{1}{\sqrt{n}}\right)\right) + r(n)\,\|f\|. \tag{3.3}$$

In this case, the term $r(n)\,\|f\|$ cannot be omitted. For example, if $f = \mathbf{1}$ and $\lambda_n \to 0$, then $\|A_n(f) - w\cdot f\| \geqslant |A_n(f)(0)| = |\lambda_n| = r(n)\,\|f\|$.

However, estimate (3.3) can be improved in every compact $[a, b] \subset]0, 1[$ by replacing $r(n)\ \|f\|$ with $r(n)\ \|f\| \sup_{a \leqslant x \leqslant b}(x^n + (1-x)^n)$.

Now, we prove the announced Voronovskaja-type formula.

THEOREM 3.4. *Let $f \in \mathscr{C}([0, 1])$ be two times differentiable at $x \in [0, 1]$. Then*

$$\lim_{n \to \infty} n(A_n(f)(x) - A_n(\mathbf{1})(x)\, f(x))$$

$$= \begin{cases} \frac{1}{2}x(1-x)\, w(x)\, f''(x) + x(1-x)\, w'(x)\, f'(x), & \text{if } 0 < x < 1, \\ 0, & \text{if } \ x = 0, 1. \end{cases} \tag{3.4}$$

Moreover, if $f \in \mathscr{C}^2([0, 1])$, then (3.4) holds uniformly on $[0, 1]$.

Proof. By Taylor's formula

$$f\left(\frac{k}{n}\right) - f(x) = \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \left(\frac{1}{2} f''(x) + \eta\left(\frac{k}{n} - x\right)\right),$$

where η is bounded and $\lim_{t \to 0} \eta(t) = 0$. Then,

$$\begin{aligned} A_n(f)(x) - f(x) \cdot A_n(\mathbf{1})(x) = {} & f'(x)(A_n(id)(x) - x \cdot A_n(\mathbf{1})(x)) \\ & + \frac{1}{2} f''(x)\, A_n((id - x \cdot \mathbf{1})^2)(x) \\ & + \sum_{k=0}^{n} \alpha_{n,k}\, x^k (1-x)^{n-k} \left(\frac{k}{n} - x\right)^2 \eta\left(\frac{k}{n} - x\right). \end{aligned}$$

Arguing as in [12, p. 22] and using the inequality $|\alpha_{n,k}| \leqslant M\binom{n}{k}$, it follows

$$\lim_{n \to \infty} n \left| \sum_{k=0}^{n} \alpha_{n,k}\, x^k (1-x)^{n-k} \left(\frac{k}{n} - x\right)^2 \eta\left(\frac{k}{n} - x\right) \right| = 0$$

(uniformly in $x \in [0, 1]$ if $f \in \mathscr{C}^2([0, 1])$).

Therefore, by Lemma 3.1(ii) and Lemma 3.2, we obtain

$$\lim_{n \to \infty} n(A_n(f)(x) - f(x) \cdot A_n(\mathbf{1})(x))$$

$$= \tfrac{1}{2} x(1-x)\, f''(x)\, w(x) + x(1-x)\, f'(x)\, w'(x).$$

The last part follows at the same manner by using the uniform convergence properties in Lemma 3.1(ii) and Lemma 3.2. ∎

Remark. **1.** If $f \in \mathscr{C}([0,1])$ is two times differentiable at $x \in]0,1[$, we can write the Voronovskaja-type formula in the following form

$$\lim_{n \to \infty} n(A_n(f)(x) - w(x) \cdot f(x))$$
$$= \tfrac{1}{2} x(1-x)\, w(x)\, f''(x) + x(1-x)\, w'(x)\, f'(x). \tag{3.5}$$

Indeed, this follows by inequality (3.2) and by

$$\limsup_{n \to \infty} n\, |f(x)|\, |A_n(\mathbf{1})(x) - w(x)|$$
$$\leqslant |f(x)| \limsup_{n \to \infty} nr(n)(x^n + (1-x)^n) = 0$$

for every $x \in]0,1[$.

Moreover, we have $\lim_{n \to \infty} n(A_n(f)(x) - w(x) \cdot f(x)) = 0$ at the endpoints if $r(n) = o(1/n)$.

2. In the case of Bernstein operators, we have $w = \mathbf{1}$ and therefore (3.4) and (3.5) reduce to the classical Voronovskaja's formula (see, e.g., [12, p. 22]).

An expressive formulation of (3.4) can be obtained for positive sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\rho_n)_{n \in \mathbb{N}}$. In this case the function w is strictly positive in $]0,1[$ and therefore if $f \in \mathscr{C}^2([0,1])$,

$$\lim_{n \to \infty} n(A_n(f)(x) - A_n(\mathbf{1}) \cdot f(x))$$
$$= \begin{cases} \dfrac{1}{2} \dfrac{x(1-x)}{w(x)} \dfrac{d}{dx}(w^2(x) f'(x)), & \text{if } \ 0 < x < 1, \\ 0, & \text{if } \ x = 0, 1. \end{cases} \tag{3.6}$$

Moreover, if $w > 0$ on $[0,1]$ (i.e., $\lambda_\infty > 0$, $\rho_\infty > 0$), then the convergence holds uniformly on $[0,1]$.

4. CONVERGENCE OF DERIVATIVES

In this brief section, we give some general results concerning the convergence of derivatives of $A_n(f)$ for a differentiable function f.

We observe that if $f \in \mathscr{C}^1([0,1])$ and $m < n$, then (see (1.8))

$$L_{m,n}(f)(x) = x(1-x)^m\, \tilde{B}_{n-m-1}(f)(x),$$

where

$$\tilde{B}_{n-m-1}(f)(x) := \sum_{k=0}^{n-m-1} \binom{n-m-1}{k} x^k (1-x)^{n-m-1-k} f\left(\frac{k+1}{n}\right).$$

Consequently

$$\begin{aligned}(L_{m,n}(f))' &= \tilde{B}_{n-m-1}(f)(x) \frac{d}{dx} (x(1-x)^m) \\ &\quad + (n-m-1)\, x(1-x)^m \sum_{k=0}^{n-m-2} \binom{n-m-2}{k} \\ &\quad \times x^k (1-x)^{n-m-2-k} \left(f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right) \\ &= \tilde{B}_{n-m-1}(f)(x) \frac{d}{dx} (x(1-x)^m) + \frac{n-m-1}{n} x(1-x)^m \\ &\quad \times \sum_{k=0}^{n-m-2} \binom{n-m-2}{k} x^k (1-x)^{n-m-2-k} f'(\xi_k), \end{aligned} \tag{4.1}$$

with $(k+1)/n \leqslant \xi_k \leqslant (k+2)/n$. Using Lagrange's theorem and the uniform continuity of f' it is easy to see that

$$\lim_{n\to\infty} (L_{m,n}(f))'(x) = \frac{d}{dx} (x(1-x)^m f(x)) \qquad \text{uniformly on} \quad [0, 1].$$

The same argument can be iterated and applied also to the operators $R_{m,n}$, B_n^l and B_n^r (see (2.14) and (2.15)), so that we can state the following result.

PROPOSITION 4.1. *For every* $f \in \mathscr{C}^k([0,1])$, *we have*

(i) $\displaystyle\lim_{n\to\infty} (L_{m,n}(f))^{(k)}(x) = \frac{d^k}{dx^k} (x(1-x)^m f(x))$ *uniformly on* $[0, 1]$;

(ii) $\displaystyle\lim_{n\to\infty} (R_{m,n}(f))^{(k)}(x) = \frac{d^k}{dx^k} (x^m(1-x) f(x))$ *uniformly on* $[0, 1]$;

(iii) $\displaystyle\lim_{n\to\infty} (B_n^l(f))^{(k)}(x) = \frac{d^k}{dx^k} ((1-x) f(x))$ *uniformly on* $[0, 1]$;

(iv) $\displaystyle\lim_{n\to\infty} (B_n^r(f))^{(k)}(x) = \frac{d^k}{dx^k} (xf(x))$ *uniformly on* $[0, 1]$.

THEOREM 4.2. *If $f \in \mathscr{C}([0,1])$ admits a derivative of order k at a point $x \in]0,1[$, then the sequence $((A_n(f))^{(k)}(x))_{n\in\mathbb{N}}$ converges to $(d^k/dx^k)(w \cdot f)(x)$. Moreover, if $f \in \mathscr{C}^k([0,1])$, then the convergence is uniform on every compact subinterval $[a,b] \subset]0,1[$.*

Proof. For brevity, we consider only the case $k=1$ and assume first $f \in \mathscr{C}^1([0,1])$. We can write

$$A_n = \sum_{m=1}^{n} \mu_m L_{m,n} + \sum_{m=1}^{n} \sigma_m R_{m,n} + \lambda_\infty B_n^l + \rho_\infty B_n^r,$$

with $\mu_n := \lambda_n - \lambda_\infty$ and $\sigma_n := \rho_n - \rho_\infty$.

Now, we have

$$\sum_{m=1}^{n} \mu_m (L_{m,n}(f))' = \sum_{m=1}^{v} \mu_m (L_{m,n}(f))' + \sum_{m=v+1}^{n} \mu_m (L_{m,n}(f))'. \qquad (1)$$

If $x \in]0,1[$, by (4.1) we obtain $|L_{m,n}(f)'(x)| \leqslant m(1-x)^{m-1} \|f\| + x(1-x)^m \|f'\|$ and hence, if $[a,b] \subset]0,1[$,

$$\sup_{a \leqslant x \leqslant b} \left| \sum_{m=v+1}^{n} \mu_m (L_{m,n}(f))'(x) \right| \leqslant \sup_{m \geqslant v} |\mu_m| \left(\|f'\| + \frac{1}{a^2} \|f\| \right)$$

and therefore the second term in the right-hand side of (1) tends uniformly to 0 on $[a,b]$ as $v \to \infty$.

If v is fixed, the first term converges uniformly to $\sum_{m=1}^{v} \mu_m (d/dx)(x(1-x)^m f(x))$. If we put $\tilde{\ell}(x) := \sum_{m=1}^{\infty} \mu_m x(1-x)^m$ then

$$\sup_{a \leqslant x \leqslant b} \left| \frac{d}{dx} (\tilde{\ell}(x) f(x)) - \sum_{m=1}^{v} \mu_m \frac{d}{dx} (x(1-x)^m f(x)) \right| \to 0 \qquad \text{as} \quad v \to \infty$$

and hence

$$\lim_{n\to\infty} \sum_{m=1}^{n} \mu_m (L_{m,n}(f))' = \frac{d}{dx} (\tilde{\ell} \cdot f)$$

uniformly in $[a,b]$. By repeating the same argument we have

$$\lim_{n\to\infty} \sum_{m=1}^{n} \sigma_m (R_{m,n}(f))' = \frac{d}{dx} (\tilde{\imath} \cdot f)$$

uniformly in $[a,b]$, where $\tilde{\imath}(x) = \sum_{m=1}^{\infty} \sigma_m x^m (1-x)$.

Hence the second part of the result immediately follows, using Proposition 4.1, (iii) and (iv). The first part can be derived in a straightforward manner (see, e.g., [12, pp. 26–27]).

Remark. In general, Theorem 4.2 is false at $x=0$ or $x=1$. For example, if $\rho_m=0$ and $\lambda_m=1/m$, then $A_n(\mathbf{1})'(x)=\sum_{m=1}^{n-1}(1/m)(1-x)^m-1$ does not converge at x = 0.

5. CONVERGENCE OF ITERATES

For arbitrary sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ of real numbers, in general we cannot ensure the convergence of the iterates of the operators A_n, $n\geqslant 1$. Here we study a case of particular interest which can be easily described.

We fix $n\geqslant 1$ and study the behavior of the sequence $(A_n^p)_{n\in\mathbb{N}}$, where as usual $A_n^1=A_n$ and $A_n^{p+1}=A_n\circ A_n^p$ for every $p\geqslant 1$. Since $A_n(f)(0)=\lambda_n f(0)$ and $A_n(f)(1)=\rho_n f(1)$ (see (2.1)) we have, for every $p\geqslant 1$,

$$A_n^p(f)(0)=\lambda_n^p f(0), \qquad A_n^p(f)(1)=\rho_n^p f(1) \tag{5.1}$$

and hence, the convergence of the sequence $(A_n^p)_{n\in\mathbb{N}}$ implies that

$$-1<\lambda_n\leqslant 1, \qquad -1<\rho_n\leqslant 1. \tag{5.2}$$

We assume $-1<\lambda_m\leqslant 1$ and $-1<\rho_m\leqslant 1$ for every $m=1, ..., n$. By (2.29) and (2.35) we have $-1<b_n\leqslant 1$ and consequently

$$0\leqslant |A_n|\,(\mathbf{1})\leqslant B_n(|b_n|)\leqslant B_n(\mathbf{1})=\mathbf{1}.$$

Hence $0\leqslant |A_n^p|\leqslant B_n$ for every $p\geqslant 1$ (we use the notation $|A_n|\,(f)=|A_n(f)|$). Moreover we observe that if $\lambda_n=1$ or $\rho_n=1$ then $\|A_n^p\|=1$ for every $p\geqslant 1$.

We have the following result.

PROPOSITION 5.1. *Assume that $-1<\lambda_m\leqslant 1$ and $-1<\rho_m\leqslant 1$ for every $m=1, ..., n$. Then the sequence $(A_n^p)_{p\in\mathbb{N}}$ is strongly convergent.*

Moreover

(1) $\lim_{p\to\infty} A_n^p(f)=0$ *for every $f\in\mathscr{C}([0,1])$ such that $f(0)=f(1)=0$.*

(2) *If $\lambda_n<1$ and $\rho_n<1$, then $(A_n^p)_{p\in\mathbb{N}}$ converges strongly to* 0.

Proof. We observe that (1) holds for Bernstein operators (see, e.g., [5, (2.5.5), p. 118]), so, if $f \in \mathscr{C}([0,1])$ satisfies $f(0)=f(1)=0$, then by the inequality

$$0 \leqslant |A_n^p(f)| \leqslant B_n^p(|f|)$$

we deduce (1) in the general case.

Now, assume $\lambda_n < 1$ and $\rho_n < 1$. Take $0 < \delta \leqslant \min\{1-|\lambda_n|,\ 1-|\rho_n|\}$; then, for every $f \in \mathscr{C}([0,1])$, $\|f\| \leqslant 1$, we easily obtain

$$0 \leqslant |A_n|\,(f)(x) \leqslant B_n(\mathbf{1})(x) - \delta(x^n + (1-x)^n) = 1 - \delta(x^n + (1-x)^n)$$

and hence $\|A_n\| < 1$, from which (2) follows.

We have only to show the convergence of $(A_n^p)_{n \in \mathbb{N}}$ in the remaining case $\max\{\lambda_n, \rho_n\} = 1$. Since $\|A_n^p\| = 1$ for every $p \geqslant 1$, the spectral radius of A_n is equal to 1. We prove that $\sigma(A_n) \cap \mathbb{D} \subset \{1\}$, where, as usual, $\sigma(A_n)$ denotes the spectrum of A_n and $\mathbb{D}$ is the closed unit disk in the plane.

For, let $f \in \mathscr{C}([0,1])$ be such that $A_n(f) = e^{i\theta} f$, $e^{i\theta} \neq 1$. Then

$$A_n(f)(0) = \lambda_n f(0) = e^{i\theta} f(0) \qquad \text{and} \qquad A_n(f)(1) = \rho_n f(1) = e^{i\theta} f(1),$$

from which $f(0)=f(1)=0$. Hence by property (1)

$$0 = \lim_{p \to \infty} A_n^p(f) = \lim_{p \to \infty} e^{ip\theta} f$$

and this yields $f = 0$.

Hence, $\sigma(A_n) \cap \mathbb{D} = \{1\}$ and 1 is a simple pole of the resolvent of A_n because $\|A_n\| = 1$. Applying [13, Proposition 3.5, p. 12], we complete the proof. ∎

Under the assumptions of Proposition 5.1, the limit operator $P_n := \lim_{p \to \infty} A_n^p$ is a projection on the eigenspace $\operatorname{Ker}(A_n - I)$ and commutes with A_n. Furthermore $\operatorname{Ker}(A_n - I)$ is contained in the space $\mathscr{P}_n$ of all polynomials with degree less or equal to n.

In particular, we have $P_n(f) = 0$ if $f(0) = f(1) = 0$.

In the following result, we show that the dimension of $\operatorname{Ker}(A_n - I)$ is just equal to the number of coefficients λ_n and ρ_n which are equal to 1.

PROPOSITION 5.2. (i) $\dim(\operatorname{Ker}(A_n - I)) = 2$ *if both* λ_n *and* ρ_n *are equal to* 1;

(ii) $\dim(\operatorname{Ker}(A_n - I)) = 1$ *if only one of* λ_n *and* ρ_n *is equal to* 1.

Proof. (i) Since $P_n(f)=0$ if f vanishes at the points 0 and 1, then $\mathrm{Ker}(A_n - I)$ is generated by $P_n(\mathbf{1}-id)$ and $P_n(id)$. But

$$P_n(\mathbf{1}-id)(0)=1,\ P_n(\mathbf{1}-id)(1)=0 \qquad \text{and} \qquad P_n(id)(0)=0,\ P_n(id)(1)=1$$

and hence the functions $P_n(\mathbf{1}-id)$ and $P_n(id)$ are linearly independent.

(ii) Suppose, for instance, $\lambda_n=1$ and $-1<\rho_n<1$. We show that $P_n(f)=0$ if $f(0)=0$. In fact by (5.1), $P_n(f)(1)=0$ and hence $P_n(f)$ vanishes at both the points 0 and 1; by property (1) of Proposition 5.1, $P_n(f)=P_n(P_n(f))=0$. In particular $P_n(id)=0$ and therefore $\mathrm{Ker}(A_n-I)$ is generated only by $P_n(\mathbf{1}-id)$ (which is non zero by (5.1) again). ▮

If we denote by $P_B:=\lim_{n\to\infty} B_n^p$ the limit projection of the iterates of the nth Bernstein operator, we have (see, e.g., [5, (2.5.5), p. 118])

$$P_B(f)(x)=(1-x)\,f(0)+xf(1)$$

for every $f\in\mathscr{C}([0,1])$ and $x\in[0,1]$ and further $|P_n|\leqslant P_B$.

In general the projection P_n has the following expression

$$P_n(f)=f(0)\ P_n(\mathbf{1}-id)+f(1)\ P_n(id) \tag{5.3}$$

for every $f\in\mathscr{C}([0,1])$.

If both λ_n and ρ_n are equal to 1, we observe that $0\leqslant|P_n|\ (\mathbf{1}-id)\leqslant P_B(\mathbf{1}-id)=\mathbf{1}-id$ and $0\leqslant|P_n|\ (id)\leqslant P_B(id)=id$.

If $\lambda_n=1$ and $\rho_n=0$ we have $0\leqslant|P_n|\ (\mathbf{1}-id)\leqslant\mathbf{1}$ and $P_n(id)=0$.

Moreover, observe that if $\lambda_m=\rho_m$ for every $m=1, ..., n$, then $A_n(\mathbf{1}-id)(1-x)=A_n(id)(x)$ and hence, in this case

$$P_n(\mathbf{1}-id)(1-x)=P_n(id)(x)$$

Related problems concerning the convergence of the iterates are considered in [9, Section 4]; however, even there the discussion is not complete and some open problems are indicated.

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